

PHYS 100C, Lecture 17

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Last lecture we learned how \vec{E}, \vec{B} transform:

$$\begin{aligned}\tilde{E}_x &= E_x \\ \tilde{E}_y &= \gamma(E_y - v B_z) \\ \tilde{E}_z &= \gamma(E_z + v B_y) \\ \tilde{B}_x &= B_x \\ \tilde{B}_y &= \gamma(B_y + \frac{v}{c^2} E_z) \\ \tilde{B}_z &= \gamma(B_z - \frac{v}{c^2} E_y)\end{aligned}$$

We would like to keep using Lorentz transform, for 4-vectors:

$$\tilde{a} = \Lambda \cdot a \quad \text{OR:}$$

$$\tilde{a}^\mu = \sum_\nu \Lambda^\mu_\nu \cdot a^\nu$$

Often \sum is dropped - summation over common index is implied (Einstein's notations)

$$\begin{pmatrix} \tilde{a}^0 \\ \tilde{a}^1 \\ \tilde{a}^2 \\ \tilde{a}^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix}$$

\uparrow \uparrow \uparrow
 \tilde{a} Λ a

Λ^μ_ν is element of μ row
 ν column

We need to represent 6
 numbers ($E_x, E_y, E_z, B_x, B_y, B_z$)
 - not possible with 4-vectors
 (4x1 matrix).

It's possible to do with
 4x4 matrix, but it's too
 much "real estate" - 16 elements

$$\begin{pmatrix} t^{00} & t^{01} & t^{02} & t^{03} \\ t^{10} & t^{11} & t^{12} & t^{13} \\ t^{20} & t^{21} & t^{22} & t^{23} \\ t^{30} & t^{31} & t^{32} & t^{33} \end{pmatrix} - 2^{nd} \text{ rank} \\ \text{tensor,} \\ 16 \text{ elements}$$

Symmetric tensor: $t^{\mu\nu} = t^{\nu\mu}$

Reduce 16 to $4 + 6 = 10$

independent variables

Antisymm. tensor: $f^{\mu\nu} = -f^{\nu\mu}$
implies: $f^{\mu\mu} = 0$

$$\begin{pmatrix} 0 & f^{01} & f^{02} & f^{03} \\ -f^{01} & 0 & f^{12} & f^{13} \\ -f^{02} & -f^{12} & 0 & f^{23} \\ -f^{03} & -f^{13} & -f^{23} & 0 \end{pmatrix} \text{ ONLY 6 elem. independent}$$

We will need to use Λ multiplication twice - once for each index:

$$\widehat{f}^{\mu\nu} = \sum_{\lambda, \epsilon} \Lambda_{\lambda}^{\mu} \Lambda_{\epsilon}^{\nu} \cdot f^{\lambda\epsilon}$$

Let's consider $\widehat{f}^{01} = \sum_{\lambda, \epsilon} \Lambda_{\lambda}^0 \Lambda_{\epsilon}^1 f^{\lambda\epsilon}$
but $\Lambda_{\lambda}^0 = 0$ FOR $\lambda = 2, 3$

$$\Lambda_{\epsilon}^1 = 0 \text{ FOR } \epsilon = 2, 3$$

$$\begin{aligned} \widehat{f}^{01} &= \sum_{\substack{\lambda=0,1 \\ \epsilon=0,1}} \Lambda_{\lambda}^0 \Lambda_{\epsilon}^1 f^{\lambda\epsilon} = \Lambda_0^0 \Lambda_0^1 f^{00} + \\ &\quad + \Lambda_0^0 \Lambda_1^1 f^{01} + \Lambda_1^0 \Lambda_0^1 f^{10} + \\ &\quad + \Lambda_1^0 \Lambda_1^1 f^{11} \end{aligned}$$

but since $f^{00} = f^{11} = 0$ (antisym.)

$$\widehat{f}^{01} = \gamma^2 f^{01} + \gamma^2 \beta^2 f^{10}$$

$$\tilde{f}^{01} = \gamma^2 f^{01} + \gamma^2 \beta^2 f^{10}$$

$$f^{10} = -f^{01} \Rightarrow$$

$$\begin{aligned} \tilde{f}^{01} &= (\gamma^2 - \gamma^2 \beta^2) f^{01} = \gamma^2 (1 - \beta^2) f^{01} \\ &= f^{01} \end{aligned}$$

\uparrow
 $\frac{1}{\gamma^2}$

Can do the same calculations
for all $f^{\mu\nu}$, and find
that f 's transform the
way $\vec{E}/c \approx \vec{B}$ transform.

Can introduce 2nd rank,
antisymm. tensors:

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

OR, ALTERNATIVELY:

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

Then:

$$\tilde{F} = \Lambda \cdot \Lambda \cdot F$$

$$\left(\text{OR } \tilde{F}^{\mu\nu} = \sum_{\lambda, \sigma} \Lambda_{\lambda}^{\mu} \Lambda_{\sigma}^{\nu} \cdot F^{\lambda\sigma} \right)$$

and

$$\tilde{G} = \Lambda \cdot \Lambda \cdot G$$

*

AND NOW, FOR SOMETHING
completely different:

$$\text{charge density: } \rho = \frac{Q}{V}$$

$$\text{current density: } \gamma = \rho \cdot u$$

$$\text{In moving frame } V = \frac{V_0}{\gamma}$$

due to Lorentz contraction
along the direction of motion.

$$\tilde{\rho} = \rho_0 \cdot \gamma$$

$$\tilde{\gamma} = \rho_0 \cdot u \cdot \gamma$$

IF we introduce $\rho \cdot c$,

then it looks just like $\rho_0 \cdot u^0$
(proper velocity), and

$$\vec{J} = \rho_0 \cdot \vec{u}^M$$

4-vector: $J^\mu = \rho_0 \cdot u^\mu = \rho_0 \begin{pmatrix} c \\ u_x \\ u_y \\ u_z \end{pmatrix}$

OR $J^\mu = \begin{pmatrix} J^0 \\ J_x \\ J_y \\ J_z \end{pmatrix}$ transforms like any other 4-vector.

Conservation of charge
(absence of "FREE LUNCH" concept)

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

$$\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = -\frac{\partial \rho}{\partial t}$$

$$\frac{\partial \rho}{\partial t} = \frac{\partial \vec{C} \cdot \vec{p}}{\partial (ct)} = \frac{\partial J^0}{\partial x^0}$$

$$\frac{\partial J^0}{\partial x^0} + \frac{\partial J^1}{\partial x^1} + \frac{\partial J^2}{\partial x^2} + \frac{\partial J^3}{\partial x^3} = 0$$

OR $\frac{\partial J^\mu}{\partial x^\mu} = 0$

(summation over $\mu=0,1,2,3$ is implied),